LOGARITHMS

❑ MELANIE'S ALLOWANCE

Melanie tells her father that she will pay for her entire college education all by herself *if* he agrees to the following plan: He gives her 2¢ on the first day of the month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on till the 30th of the month. After that month – no more money. Dad (who was a philosophy major) thinks this is a great money-saving idea and accepts Melanie's proposal.

Day	# Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
⋮	⋮
30	1,073,741,824 (almost $11 million)

In the chart, we have calculated Melanie's earnings for each of the first 10 days of the month; then we cut to the chase and calculated the amount for the 30th day. Take your calculator and verify each of the 11 amounts of money in the table's second column. You should start at the 10th day, and then double the 1,024¢ to get 2,048¢ for the 11th day, and keep doubling until you get to the 30th day.

Now let's come up with a direct formula that computes the money earned from the day of the month, <u>without knowing all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day.

[For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$.]

Now, for the 30th day, calculate 2^{30}, and you should get **1,073,741,824** (which, remember, is the number of pennies Melanie gets on day 30; you can divide this number of pennies by 100, and then round off to see that it's equivalent to about 10.7 million dollars!

Homework

1. a. How many pennies will Melanie earn on the 15th day of the month?

   b. How many pennies will Melanie earn on the 31st of the month if Dad agrees to extend the plan that far?

   c. On which day of the month did Melanie earn half of what she earned on the 30th day?

2. Now for a little practice in the concept to be covered in this chapter. I'll give you a penny amount, and you tell me which day of the month Melanie earned that amount of money.

   a. 512¢ b. 4,096¢ c. 1,048,576¢ d. 33,554,432¢

3. Similar question, but a little more abstract: I'll give you a number, and you tell me the *exponent* that **2** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 2,048, then you say "11" because $2^{11} = 2{,}048$.

   a. 2 b. 256 c. 64 d. 1
   e. 8,192 f. 131,072 g. 524,288 h. 1/2

4. Another question like #3, but now when I give you the number, you tell me the *exponent* that **10** would have to be raised to, in

order to get the number I gave you. For example, if I give you the number 1,000, then you say "3" because $10^3 = 1{,}000$.

a. 100 b. 10,000 c. 10 d. 1

e. 100,000 f. 1,000,000 g. 1 billion h. 1 googol

❑ The Meaning of a Log

A ***log*** (short for ***logarithm***) is an exponent. It's the exponent to which one number (called the base) must be raised to get a specified number. This definition is so far off in the clouds that we need an example right now!

> *logarithm*
> from the Greek:
> *logos* = reckoning
> *arithmos* = number

For our first example, to calculate

$$\log_{10}(1000)$$ [read: "log, base 10, of 1000" or "log of 1000, base 10]

we ask ourselves, "10 raised to *what power* equals 1000?" In other words, 10 to the "what" equals 1000? The answer is 3, since $10^3 = 1000$. Therefore,

$$\log_{10}(1000) = \mathbf{3}$$ [log, base 10, of 1000 is 3.]

For a second example, let's analyze $\log_2 32$, which asks us, "2 raised to the "what" equals 32?" Well, 2 to the 5th power equals 32, and so

$$\log_2(32) = \mathbf{5}$$ [log, base 2, of 32 is 5.]

Our third example will describe a log a little differently: If you can fill in the box in the equation

$$4^{\square} = 16$$

then you have found the "log, base 4, of 16," which is 2. That is,

$\log_4(16) = \mathbf{2}$

Notation: A log is a function, so notation like $\log_2(128)$ certainly makes sense, just like when we use parentheses in function notation: $f(x)$. But if it's clear what we're taking the log of, we don't really need the parentheses; so, for example, $\log_2(128)$ is simply written $\log_2 128$. [Although in computer programming, the parentheses are required.]

Summary:

$\log_{10} 1000 = 3$ because $10^{\mathbf{3}} = 1000$

$\log_2 32 = 5$ because $2^{\mathbf{5}} = 32$

$\log_4 16 = 2$ because $4^{\mathbf{2}} = 16$

This is really abstract, isn't it? Let's get right to some homework.

Homework

5. To find $\log_5 25$, which is read "log, base 5, of 25," ask yourself, "5 raised to what power equals 25?" $5^{\square} = 25$

6. To find $\log_2 8$, which is read "log, base 2, of 8," ask yourself, "2 raised to what power equals 8?" $2^{\square} = 8$

7. To find $\log_9 9$, which is read "log, base 9, of 9," ask yourself, "9 raised to what power equals 9?" $9^{\square} = 9$

8. To find $\log_{17} 1$, which is read "log, base 17, of 1," ask yourself, "17 raised to what power equals 1?" $17^{\square} = 1$

9. To find $\log_{100} 10$, which is read “log, base 100, of 10,” ask yourself, “100 raised to what power equals 10?” $100^{\square} = 10$

10. To find $\log_6\left(\frac{1}{6}\right)$, which is read “log, base 6, of 1/6,” ask yourself, “6 raised to what power equals $\frac{1}{6}$?” $6^{\square} = \frac{1}{6}$

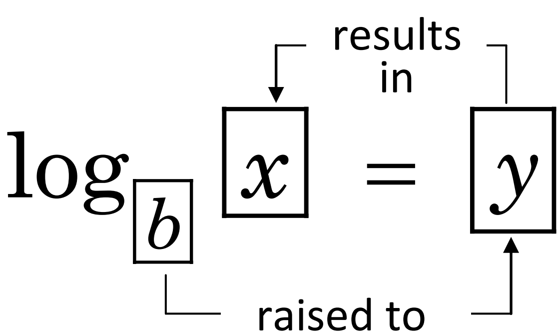
❑ *THE OFFICIAL DEFINITION OF LOGARITHM*

$$\log_b x = y \quad \text{means} \quad b^y = x$$

The notation “$\log_b x$” is read either

“log, base *b*, of *x*” or “log of *x*, base *b*”

Here’s another way to visualize the meaning of logarithm:



EXAMPLE 1:

A. $\log_{10} 10{,}000 = 4$ Why? Because $10^4 = 10{,}000$

B. $\log_e e^2 = \mathbf{2}$ because $e^2 = e^2$

C. $\log_{17} 17 = \mathbf{1}$ because $17^1 = 17$

D. $\log_e 1 = \mathbf{0}$ because $e^0 = 1$

E. $\log_{25} 5 = \mathbf{\frac{1}{2}}$ because $25^{1/2} = \sqrt{25} = 5$

F. $\log_{64} 4 = \mathbf{\frac{1}{3}}$ because $64^{1/3} = \sqrt[3]{64} = 4$

G. $\log_8 4 = \mathbf{\frac{2}{3}}$ because $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

H. $\log_{13}\left(\frac{1}{13}\right) = -\mathbf{1}$ because $13^{-1} = \frac{1}{13}$

I. $\log_6\left(\frac{1}{36}\right) = -\mathbf{2}$ because $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

J. $\log_{49}\left(\frac{1}{7}\right) = -\mathbf{\frac{1}{2}}$ because $49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

K. $\log_{1/2}\left(\frac{1}{8}\right) = \mathbf{3}$ because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

<u>EXAMPLE 2:</u>

A. $\log_b b = \mathbf{1}$ since $b^1 = b$

B. $\log_b 1 = \mathbf{0}$ since $b^0 = 1$

C. $\log_b \frac{1}{b} = -1$ since $b^{-1} = \frac{1}{b}$

D. $\log_b \sqrt{b} = \frac{1}{2}$ since $b^{1/2} = \sqrt{b}$

E. $\log_b \sqrt[n]{b} = \frac{1}{n}$ since $b^{1/n} = \sqrt[n]{b}$

E. $\log_b b^n = \boldsymbol{n}$ since $b^n = b^n$

Homework

Find the value of each log:

11.	a. $\log_{10} 100$	b. $\log_5 125$	c. $\log_8 64$	d. $\log_2 64$
12.	a. $\log_e e^5$	b. $\log_b b^2$	c. $\log_{\sqrt{2}} \sqrt{2}$	d. $\log_L L$
13.	a. $\log_{10} 1$	b. $\log_e 1$	c. $\log_{\sqrt[5]{99}} 1$	d. $\log_b 1$
14.	a. $\log_{36} 6$	b. $\log_{49} 7$	c. $\log_{144} 12$	d. $\log_b \sqrt{b}$
15.	a. $\log_5 \left(\frac{1}{5}\right)$	b. $\log_e \left(\frac{1}{e}\right)$	c. $\log_{1/e} 1$	d. $\log_n \left(\frac{1}{n}\right)$
16.	a. $\log_Q Q^n$	b. $\log_x 1$	c. $\log_{2.3} 2.3$	d. $\log_9 81$
17.	a. $\log_8 2$	b. $\log_{64} 4$	c. $\log_{125} 5$	d. $\log_a \sqrt[3]{a}$

❑ *CALCULATING LOGS*

The homework problems above were designed to be solved by inspection (with a little experimentation and insight). Some logs aren't easy to do that way. So now we present a longer, but more systematic way, of evaluating logs by solving certain exponential equations.

<u>EXAMPLE 3:</u> **Calculate:** $\mathbf{\log_{27} 9}$

Solution: Let's give our log expression a name — call it y. Now we can write an equation:

$$\log_{27} 9 = y$$

The definition of log shows us how we can convert our log equation into an exponential equation:

$$27^y = 9$$

And now we solve for y. The chapter *Exponential Equations* showed us how:

$$27^y = 9 \Rightarrow (3^3)^y = 3^2 \Rightarrow 3^{3y} = 3^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

But y was the name we gave to the original log problem. So we can conclude that

$$\boxed{\log_{27} 9 = \frac{2}{3}}$$

To **check** our result, we can raise 27 to the $\frac{2}{3}$ power and make sure it comes out 9:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9 \quad \checkmark$$

Homework

Find the value of each log:

18. $\log_{64} 16$	19. $\log_{25}\left(\frac{1}{5}\right)$	20. $\log_{16} 8$
21. $\log_{27}\left(\frac{1}{9}\right)$	22. $\log_{8} 16$	23. $\log_{32}\left(\frac{1}{2}\right)$
24. $\log_{1/3}\left(\frac{1}{9}\right)$	25. $\log_{1/4} 64$	26. $\log_{9} 1$

27. $\log_{\pi} \pi^5$

28. $\log_{\pi} \sqrt{\pi}$

29. $\log_4\left(\frac{1}{8}\right)$

30. $\log_5\left(\frac{1}{25}\right)$

31. $\log_3\left(\frac{1}{\sqrt{3}}\right)$

32. $\log_4\left(\frac{1}{64}\right)$

33. $\log_2\left(\frac{1}{128}\right)$

34. $\log_{10} \sqrt{10}$

35. $\log_{10} \sqrt[3]{100}$

36. $\log_{10}\left(\frac{1}{100}\right)$

37. $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$

38. $\log_{10}\left(\frac{1}{\sqrt{1000}}\right)$

❑ *THE pH SCALE FOR ACIDS AND BASES*

One of the uses of logs is defining the **pH scale** for acids and bases. The official definition of the pH of a substance is the negative **logarithm** (base 10) of the hydrogen ion concentration of the substance. Acids (like lemonade) have a pH smaller than 7, while bases (like Drano) have a pH higher than 7. The pH of pure water is a neutral 7. [The word alkali is another term for base.]

We'll use the official chemistry notation for the hydrogen ion concentration, $[H^+]$, which has the units of moles/liter. It is not necessary to understand any of the chemistry — indeed, the math takes care of everything. Devised by a biochemist while working on the brewing of beer, the **pH** of a substance is defined to be the negative logarithm (base 10) of the hydrogen ion concentration:

$$\text{pH} = -\log_{10}[H^+]$$

New Notation and Calculator Hints:

> Logs are also used in the definition of the *Richter Scale* for earthquakes, and for the *decibel scale* for measuring the loudness of sound.

1. On a TI-30, to enter a number like 1.6×10^{-13}, first press 1.6, then press the "EE" button, and then press 13, and last the +/– key. Your display should then look something like 1.6 $^{-13}$ (the base of 10 is understood).

2. To find the "log, base 10, of 1000," $\log_{10} 1000$, enter 1000 into your calculator and then press the ***log*** button. You should, of course, get an answer of 3. On newer calculators, try pressing the *log* button first, followed by 1000. Using 10 as a base for logs is so "common" that it is officially referred to as the **common log**, and we dispense with writing the base, 10 — it's "understood":

$$\log_{10} x \text{ is written } \log x$$

3. [Skip the following if you've never come across the number e.] A base of ***e*** is so important and occurs so "naturally" in the universe that "log, base ***e***" also gets its own name and notation:

$$\log_e x \text{ is written } \ln x$$

You read "ln x" either as "el en x" or "el en of x" or "the **natural log** of x." Teachers will many times write it on the whiteboard in cursive:

ln x (*l* for **l**og, *n* for **n**atural)

<u>EXAMPLE 4</u>: **A sample of orange juice has a hydrogen ion concentration of 2.9×10^{-4} moles/liter. Find the pH of the orange juice.**

Solution: According to the definition,

$$\text{pH} = -\log_{10}[\text{H}^+] = -\log(2.9\times10^{-4}) \approx -(-3.54) = 3.54$$

According to the text next to the lemonade stand above, this should mean that orange juice is an acid, as indeed it is (the sour taste is one clue). Thus, the pH of the orange juice is

$$\boxed{3.54}$$

Homework

39. The hydrogen ion concentration of household ammonia is 1.26×10^{-12} moles/liter. Find the pH of the ammonia. Is it an acid or a base?

40. Pure water has a hydrogen ion concentration of 1.0×10^{-7} moles/liter. Prove that water has a neutral pH of 7.

41. Find the pH of each substance given its molarity:

    a. 1.3×10^{-2} moles/L
    b. 2.8×10^{-6} moles/L
    c. 0.3×10^{-10} moles/L
    d. 9.2×10^{-12} moles/L
    e. 5.9×10^{-7} moles/L
    f. 8.0×10^{-1} moles/L

Review Problems

42. a. $\log_{10} 1{,}000{,}000 =$ b. $\log_3 3^7 =$ c. $\log_e e =$

d. $\log_7 1 =$ e. $\log_{16} 4 =$ f. $\log_9\left(\frac{1}{9}\right) =$

g. $\log_5\left(\frac{1}{25}\right) =$ h. $\log_{36}\left(\frac{1}{6}\right) =$ i. $\log_2 512 =$

43. a. $\log_a a =$ b. $\log_c 1 =$ c. $\log_7 7^N =$

44. The hydrogen ion concentration of an unknown liquid is 3.4×10^{-11} moles/L. Find the pH of the liquid.

45. What kind of music do they play in a lumber camp?

Solutions

1. a. 32,768 b. 2,147,483,648 c. I'm not gonna tell you that.

2. a. day 9 b. day 12 c. day 20 d. day 25

3. a. 1 b. 8 c. 6 d. 0 e. 13 f. 17 g. 19 h. –1

4. a. 2 b. 4 c. 1 d. 0 e. 5 f. 6 g. 9 h. 100

5. $5^{\boxed{?}} = 25$; since $5^2 = 25, \log_5 25 = 2$.

6. $2^{\boxed{?}} = 8$; since $2^3 = 8, \log_2 8 = 3$.

7. 1 **8**. 0

9. $100^{\boxed{?}} = 10$; since $100^{1/2} = \sqrt{100} = 10, \log_{100} 10 = \frac{1}{2}$.

10. –1

11. a. 2 b. 3 c. 2 d. 6 **12**. a. 5 b. 2 c. 1 d. 1

13. a. 0 b. 0 c. 0 d. 0

14. a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{2}$

15. a. –1 b. –1 c. 0 d. –1

16. a. n b. 0 c. 1 d. 2

17. a. $\frac{1}{3}$ b. $\frac{1}{3}$ c. $\frac{1}{3}$ d. $\frac{1}{3}$

18. Let $y = \log_{64} 16 \Rightarrow 64^y = 16 \Rightarrow (4^3)^y = 4^2$

$\Rightarrow 4^{3y} = 4^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$

19. $-\frac{1}{2}$ **20**. $\frac{3}{4}$ **21**. $-\frac{2}{3}$ **22**. $\frac{4}{3}$ **23**. $-\frac{1}{5}$ **24**. 2

25. –3 **26**. 0 **27**. 5 **28**. $\frac{1}{2}$ **29**. $-\frac{3}{2}$

30. –2 **31**. $-\frac{1}{2}$ **32**. –3 **33**. –7 **34**. $\frac{1}{2}$

35. $\frac{2}{3}$ **36**. –2 **37**. $-\frac{1}{2}$ **38**. $-\frac{3}{2}$

39. pH = 11.9; it's a base (an alkali), since its pH is greater than 7.

40. $\text{pH} = -\log[\text{H}^+] = -\log(1.0\times10^{-7}) = -(-7) = 7$

41. a. 1.89 b. 5.55 c. 10.52 d. 11.04 e. 6.23 f. 0.10

42. a. 6 b. 7 c. 1 d. 0 e. $\frac{1}{2}$ f. –1 g. –2 h. $-\frac{1}{2}$ i. 9

43. a. 1 b. 0 c. N

44. 10.47 (an alkali, or base)

45. I just can't give the answer away.

"Learning is not the product of teaching.

Learning is the product of the activity of learners."

– John Holt, American psychologist and educator